Area & Volume

Maths Past Exam Questions

Marking Schemes

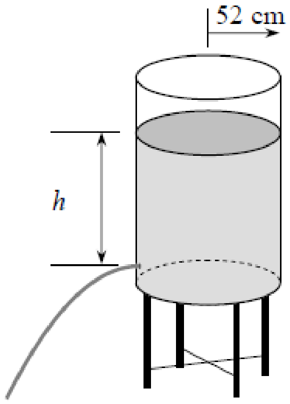
Higher Level

2012

Project Maths - Paper One - Section B - Question 7

Question 7 **(50 marks)**

An open cylindrical tank of water has a hole near the bottom. The radius of the tank is 52 cm. The hole is a circle of radius 1 cm. The water level gradually drops as water escapes through the hole.



Over a certain 20-minute period, the height of the surface of the water is given by the formula

$$h = \left(10 - \frac{t}{200}\right)^2$$

where h is the height of the surface of the water, in cm, as measured from the centre of the hole,
and t is the time in seconds from a particular instant $t = 0$.

(a) What is the height of the surface at time $t = 0$?

$$h(0) = 10^2 = 100 \text{ cm.}$$

(b) After how many seconds will the height of the surface be 64 cm?

$$\left(10 - \frac{t}{200}\right)^2 = 64$$

$$10 - \frac{t}{200} = 8 \qquad (\text{since } t > 0)$$

$$t = 400$$

Answer: 400 seconds.

(c) Find the rate at which the **volume** of water in the tank is decreasing at the instant when the height is 64 cm.
Give your answer correct to the nearest cm^3 per second.

$$V = \pi r^2 h = \pi(52)^2 h = 2704\pi h.$$

$$\frac{dV}{dt} = 2704\pi \frac{dh}{dt}$$

$$= 2704\pi\left(\tfrac{-2}{25}\right) = -216{\cdot}32\pi$$

$$\frac{dh}{dt} = 2\left(10 - \frac{t}{200}\right)\frac{-1}{200}$$

$$\left.\frac{dh}{dt}\right|_{t=400} = \frac{-2}{25}$$

$\therefore$ Volume is decreasing at $216{\cdot}32\pi \text{ cm}^3\text{ s}^{-1} \approx 680 \text{ cm}^3\text{ s}^{-1}$.

(d) The rate at which the volume of water in the tank is decreasing is equal to the speed of the water coming out of the hole, multiplied by the area of the hole. Find the speed at which the water is coming out of the hole at the instant when the height is 64 cm.

$$\frac{dV}{dt} = Av$$

$$216{\cdot}32\pi = \pi 1^2 v$$

$$v = 216{\cdot}32 \text{ cm s}^{-1}$$

(e) Show that, as t varies, the speed of the water coming out of the hole is a constant multiple of $\sqrt{h}$.

$$v = -\frac{1}{\pi}\frac{dV}{dt}$$

$$= -\frac{1}{\pi}2704\pi\frac{dh}{dt}$$

$$= 27{\cdot}04\left(10 - \frac{t}{200}\right)$$

$$= 27{\cdot}04\sqrt{h}$$

which is a constant multiple of $\sqrt{h}$

(f) The speed, in centimetres per second, of water coming out of a hole like this is known to be given by the formula

$$v = c\sqrt{1962h}$$

where c is a constant that depends on certain features of the hole.
Find, correct to one decimal place, the value of c for this hole.

$$c\sqrt{1962} = 27{\cdot}04$$

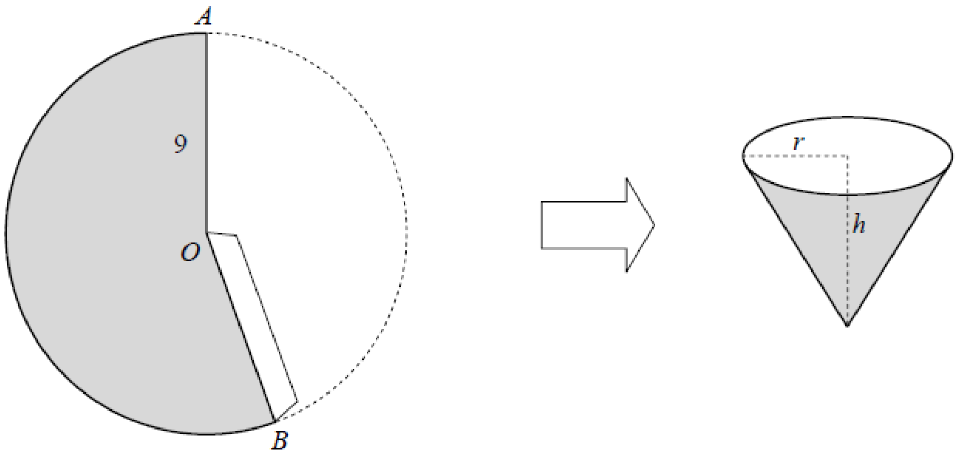
$$c \approx 0{\cdot}6$$

Q8

Question 8 **(50 marks)**

A company uses waterproof paper to make disposable conical drinking cups. To make each cup, a sector AOB is cut from a circular piece of paper of radius 9 cm. The edges AO and OB are then joined to form the cup, as shown.

The radius of the rim of the cup is r, and the height of the cup is h.



(a) By expressing r^2 in terms of h, show that the capacity of the cup, in cm^3, is given by the formula

$$V = \frac{\pi}{3} h\left(81 - h^2\right).$$

$$r^2 + h^2 = 9^2$$

$$r^2 = 81 - h^2$$

$$V = \frac{1}{3}\pi r^2 h$$

$$= \frac{\pi h}{3}\left(81 - h^2\right)$$

(b) There are two positive values of h for which the capacity of the cup is $\frac{154\pi}{3}$.

One of these values is an integer.
Find the two values.
Give the non-integer value correct to two decimal places.

$$\frac{\pi h}{3}\left(81-h^2\right)=\frac{154\pi}{3}$$

$$h\left(81-h^2\right)=154$$

$$h^3-81h+154=0$$

Integer root is a factor of 154 $\Rightarrow \in \{1, 2, 7, 14, 11, 22, 77, 154\}$

$h=1$ is not a solution; $h=2$ is a solution.

$$\begin{array}{r|l} & h^2+2h-77 \\ \hline h-2 & h^3+0h^2-81h+154 \\ & \underline{h^3-2h^2} \\ & 2h^2-81h \\ & \underline{2h^2-4h} \\ & -77h+154 \\ & \underline{-77h+154} \\ & 0 \end{array}$$

$$h^2+2h-77=0$$

$$(h+1)^2-78=0$$

$$h=-1\pm\sqrt{78}$$

Positive solutions are $h=2,\quad h\approx 7.83$

(c) Find the maximum possible volume of the cup, correct to the nearest cm^3.

$$V = \frac{\pi h}{3}\left(81 - h^2\right), \qquad h \in [0,9]$$

$$= \frac{\pi}{3}\left(81h - h^3\right)$$

$$\frac{dV}{dh} = \pi\left(27 - h^2\right)$$

Local max/min when $\frac{dV}{dh} = 0 \;\Rightarrow\; h = \sqrt{27}$. (Clearly a max., since $V(0) = V(9) = 0$.)

$$V_{max} = \pi\left(27\sqrt{27} - 9\sqrt{27}\right) = 18\sqrt{27}\pi \approx 294 \text{ cm}^3$$

(d) Complete the table below to show the radius, height, and capacity of each of the cups involved in parts **(b)** and **(c)** above.
In each case, give the radius and height correct to two decimal places.

	cups in part **(b)**		cup in part **(c)**
radius (r)	8·77 cm	4·43 cm	7·35 cm
height (h)	2 cm	7·83 cm	5·20 cm
capacity (V)	$\frac{154\pi}{3} \approx 161$ cm^3	$\frac{154\pi}{3} \approx 161$ cm^3	294 cm^3

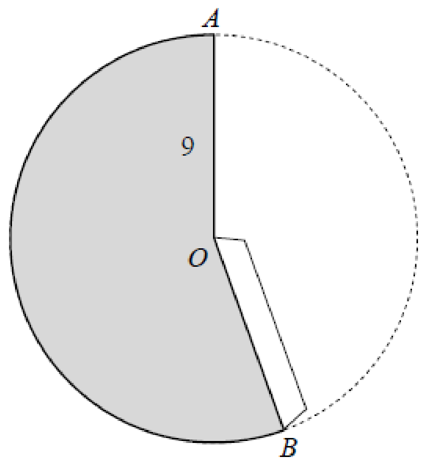
(e) In practice, which one of the three cups above is the most reasonable shape for a conical cup? Give a reason for your answer.

The middle one (radius 4·43 cm, height 7·83 cm).
The others are much too wide and shallow to hold.

(f) For the cup you have chosen in part **(e)**, find the measure of the angle *AOB* that must be cut from the circular disc in order to make the cup.
Give your answer in degrees, correct to the nearest degree.

Circumference of rim $= 2\pi r \approx 8{\cdot}86\pi \approx 27{\cdot}86$ cm.

$$\theta = \frac{l}{r} = \frac{27{\cdot}86}{9} = 3{\cdot}096 \text{ rad} \approx 177°$$



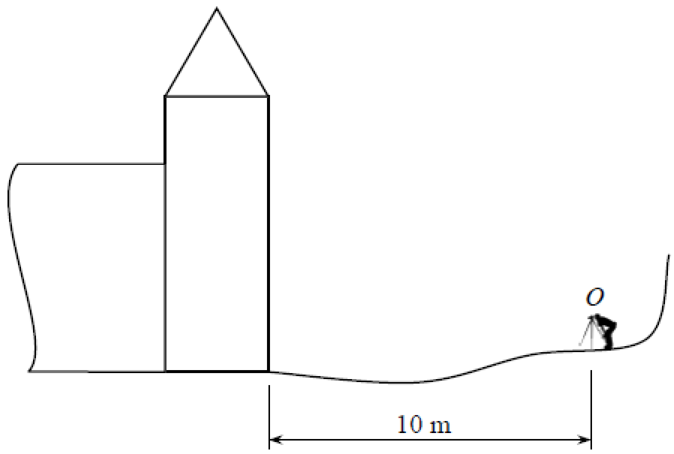
2011

Paper 2 – Q8

(a) A tower that is part of a hotel has a square base of side 4 metres and a roof in the form of a pyramid. The owners plan to cover the roof with copper. To find the amount of copper needed, they need to know the total area of the roof.

A surveyor stands 10 metres from the tower, measured horizontally, and makes observations of angles of elevation from the point O as follows:

The angle of elevation of the top of the roof is 46°.
The angle of elevation of the closest point at the bottom of the roof is 42°.
The angle of depression of the closest point at the bottom of the tower is 9°.



(i) Find the vertical height of the roof.

In the solid triangle:

$$\tan 42° = \frac{y}{10}$$

$$y = 10 \tan 42$$

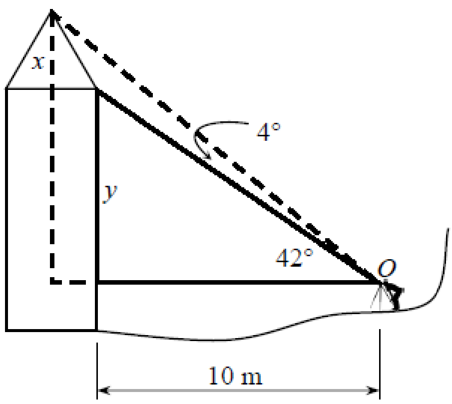
$$y = 9{\cdot}004$$

In the dashed triangle:

$$\tan 46° = \frac{x+y}{12}$$

$$x + y = 12 \tan 46$$

$$x = 12{\cdot}426 - 9{\cdot}004 = 3{\cdot}42 \text{ m}$$

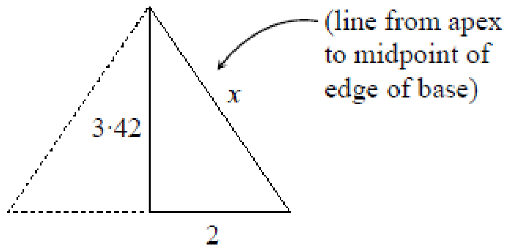


(ii) Find the total area of the roof.

$x = \sqrt{3{\cdot}42^2 + 2^2}$

$x = 3{\cdot}964$

Area = $4\left(\frac{1}{2}(4)(3{\cdot}964)\right) = 31{\cdot}71\ \text{m}^2$



(iii) If all of the angles observed are subject to a possible error of ±1°, find the range of possible areas for the roof.

Maximum possible area of roof given by:

Angle of elevation of bottom = 41°

Angle of elevation of top = 47°

$\therefore$ Height = $12\tan 47 - 10\tan 41 = 4{\cdot}18$ m

$x = \sqrt{4{\cdot}18^2 + 2^2} = 4{\cdot}634$ m

Area = $37{\cdot}07\ \text{m}^2$

Minimum possible area of roof given by:

Angle of elevation of bottom = 43°

Angle of elevation of top = 45°

$\therefore$ Height = $12\tan 45 - 10\tan 43 = 2{\cdot}675$ m

$x = \sqrt{2{\cdot}675^2 + 2^2} = 3{\cdot}34$ m

Area = $26{\cdot}72\ \text{m}^2$

$26{\cdot}72\ \text{m}^2 \leq$ area of roof $\leq 37{\cdot}07\ \text{m}^2$.

(b) Twenty five students each measure and record a particular angle of elevation, in degrees, each using his or her own home-made clinometer. The results are as follows:

24	20	22	15	70
15	16	15	16	15
18	16	21	21	73
16	20	12	18	20
18	18	14	22	18

(i) Find what you consider to be the best estimate of the true value of the angle, explaining your reasoning.

The data include two extreme outliers: 70 and 73. These probably arise from students reading the complementary angle $(90° - \theta)$ on their clinometers. The most reasonable course of action is either to correct them or discard them.

- Change 70 to 20 and 73 to 17. Then the mean = 17·88° is the best estimate for the true value of the angle.
- Exclude 70 and 73 as outliers. Then the mean = 17·83°.
- Use the median, as it less influenced by outliers. Median = 18, (irrespective of treatment of outliers).

(ii) Based on previous experience, a teacher has claimed that, in these circumstances, half of all students will measure the angle correctly to within two degrees. Taking these students to be a simple random sample, and assuming the true value of the angle is the one you calculated in part **(i)**, is there sufficient evidence to reject the teacher's claim at the 5% level of significance?

H_0 : Half of all students will measure the angle correctly to within 2°.

95% margin of error for sample of size 25: $\frac{1}{\sqrt{25}} = \frac{1}{5} = 0{\cdot}2$.

Reject H_0 if sample proportion lies outside $0{\cdot}5 \pm 0{\cdot}2$.

Taking 17·88° (or 17·83°) as the best estimate, then 8 students measured correctly to within 2°

Sample proportion: $\frac{8}{25} = 0{\cdot}32$. $\therefore$ Do not reject H_0. Answer: No.

Or

Taking 18° as the best estimate, then 12 students measured correctly to within 2°

Sample proportion: $\frac{12}{25} = 0{\cdot}48$. $\therefore$ Do not reject H_0. Answer: No.

Or

Confidence interval at 5% level of significance = $0{\cdot}32 \pm 0{\cdot}2$ or $0{\cdot}48 \pm 0{\cdot}2$. In either case, this interval contains the hypothesised proportion of 0·5, so do not reject H_0.

Or

Exact (binomial) test with $p = 0{\cdot}5$, $n = 25$ yields acceptance for r from 8 to 17 inclusive.