Geometry

Maths Past Exam Questions

Marking Schemes

Higher Level

2013

Paper 2 – Project Maths – Section A - Q5

Question 5 **(25 marks)**

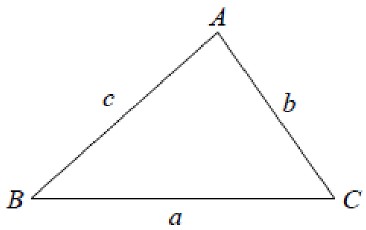
(a) In a triangle ABC, the lengths of the sides are a, b and c.
Using a formula for the area of a triangle, or otherwise, prove that

$$\frac{a}{\sin\angle A}=\frac{b}{\sin\angle B}=\frac{c}{\sin\angle C}.$$

$$\frac{1}{2}ac\sin\angle B=\frac{1}{2}ab\sin\angle C$$

Divide by $\frac{1}{2}abc$

$$\frac{\sin\angle B}{b}=\frac{\sin\angle C}{c}\Rightarrow\frac{b}{\sin\angle B}=\frac{c}{\sin\angle C}$$



OR

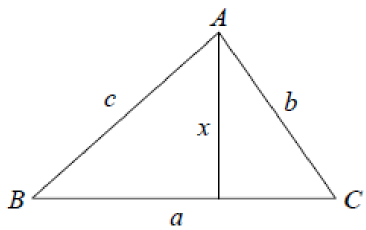
Case 1

$\sin\angle B=\frac{x}{c}$ $\qquad$ $\sin\angle C=\frac{x}{b}$

$x=c\sin\angle B$ $\qquad$ $x=b\sin\angle C$

$b\sin\angle C=c\sin\angle B$

$$\frac{b}{\sin\angle B}=\frac{c}{\sin\angle C}$$



Case 2

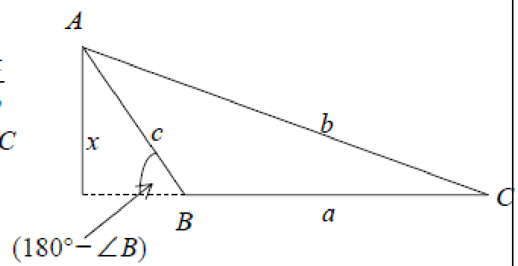
$\sin(180^\circ-\angle B)=\frac{x}{c}$ $\qquad$ $\sin\angle C=\frac{x}{b}$

$x=c\sin(180^\circ-\angle B)$ $\qquad$ $x=b\sin\angle C$

$x=c\sin\angle B$

$b\sin\angle C=c\sin\angle B$

$$\frac{b}{\sin\angle B}=\frac{c}{\sin\angle C}$$



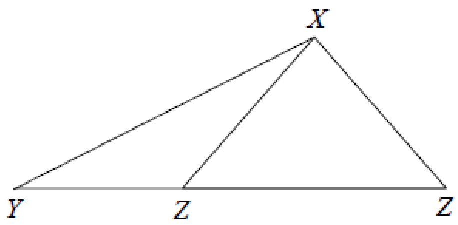
(b) In a triangle XYZ, $|XY|=5$ cm, $|XZ|=3$ cm and $|\angle XYZ|=27°$.

(i) Find the two possible values of $|\angle XZY|$. Give your answers correct to the nearest degree.

$$\frac{3}{\sin 27^\circ}=\frac{5}{\sin\angle Z}\quad\Rightarrow\quad\sin\angle Z=\frac{5\sin 27^\circ}{3}=0\cdot 756$$

$$\Rightarrow\quad|\angle Z|=49^\circ\quad\text{or}\quad|\angle Z|=131^\circ$$

(ii) Draw a sketch of the triangle XYZ, showing the two possible positions of the point Z.



(c) In the case that $|\angle XZY| < 90°$, write down $|\angle ZXY|$, and hence find the area of the triangle XYZ, correct to the nearest integer.

$|\angle ZXY| = 180° - (27° + 49°) = 104°$

$\Delta = \frac{1}{2}ab\sin C = \frac{1}{2}(5)(3)\sin 104° = 7 \cdot 27 = 7 \text{ cm}^2$

2012

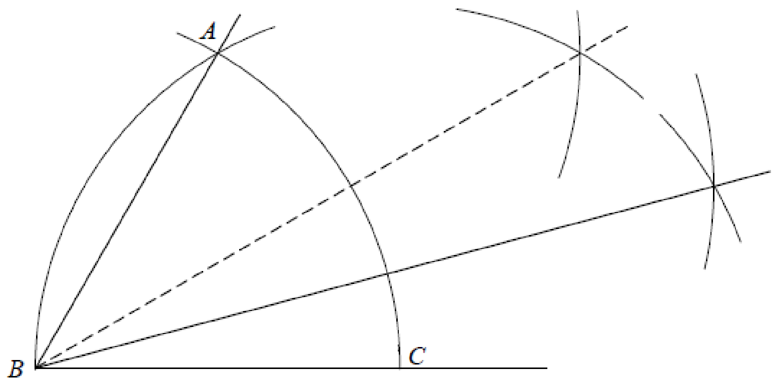
Paper 2 – Project Maths – Section A – Q6

Question 6 **(25 marks)**

Answer **either** 6A **or** 6B.

Question 6A

(a) **(i)** Given the points B and C below, construct, without using a protractor or setsquare, a point A such that $|\angle ABC| = 60°$.

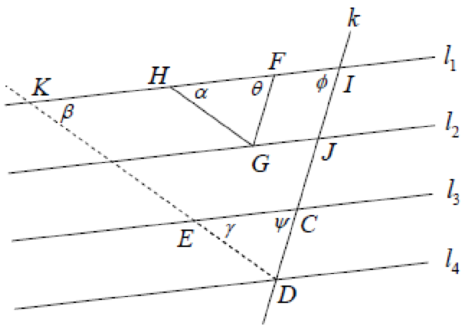


(ii) Hence construct, on the same diagram above, and using a compass and straight edge only, an angle of 15°.

Bisect 60° to get 30°; bisect again to get 15° (as shown above)
OR
Construct a right angle and use it to construct 45° and combine with 60° to get 15°.

(b) In the diagram, l_1, l_2, l_3, and l_4 are parallel lines that make intercepts of equal length on the transversal k. FG is parallel to k, and HG is parallel to ED.

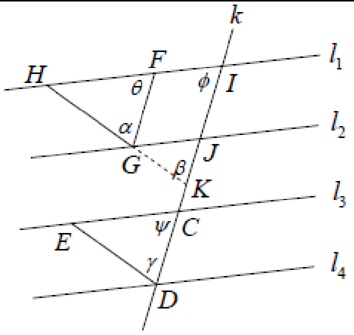
Prove that the triangles ΔCDE and ΔFGH are congruent.



$\lvert CD\rvert = \lvert IJ\rvert$	(given)
$= \lvert FG\rvert$	(opposite sides of parallelogram
$\theta = \phi = \psi$	(corresponding angles)
$\alpha = \beta = \gamma$	(corresponding angles)
$\Rightarrow \lvert\angle HGF\rvert = \lvert\angle EDC\rvert$	
$\therefore \Delta CDE \equiv \Delta FGH$	(ASA)

OR

$\lvert CD\rvert = \lvert IJ\rvert$	(given)
$= \lvert FG\rvert$	(opposite sides of parallelogram
$\theta = \phi = \psi$	(corresponding angles)
$\alpha = \beta = \gamma$	(corresponding angles)
$\therefore \Delta CDE \equiv \Delta FGH$	(ASA)

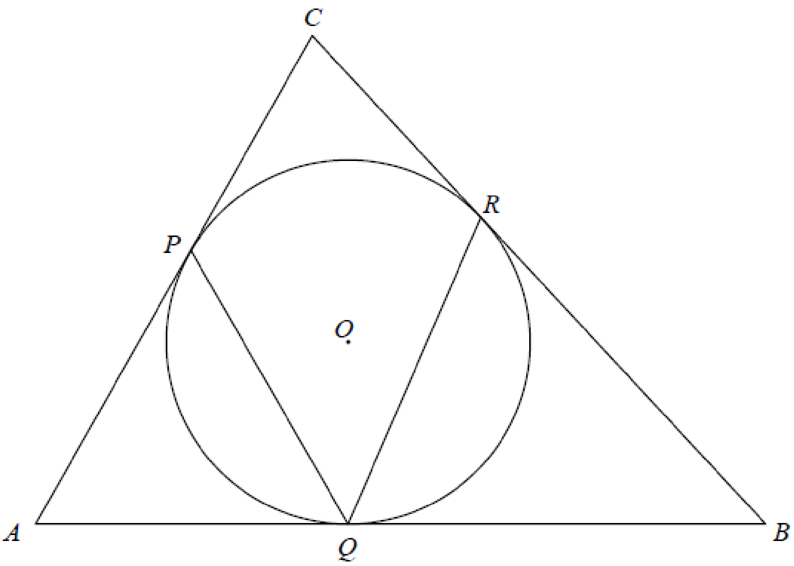


OR

Question 6B

The incircle of the triangle ABC has centre O and touches the sides at P, Q and R, as shown.

Prove that $\lvert\angle PQR\rvert = \frac{1}{2}\left(\lvert\angle CAB\rvert + \lvert\angle CBA\rvert\right)$.



$\lvert\angle OQA\rvert = \lvert\angle OPA\rvert = 90^\circ$	(radius $\perp$ tangent)
$\therefore$ O, Q, A, P are concyclic.	
$\lvert\angle OQP\rvert = \lvert\angle OAP\rvert$	(standing on same arc OP)
$= \frac{1}{2}\lvert\angle PAQ\rvert$	(since [AO is the bisector of $\angle PAQ$)
Similarly, $\lvert\angle OQR\rvert = \frac{1}{2}\lvert\angle QBR\rvert$	
Adding these two gives the required result.	

OR

$\lvert\angle OPC\rvert = \lvert\angle ORC\rvert = 90°$	(radius $\perp$ tangent)
$\therefore \lvert\angle PBR\rvert = 180° - \lvert\angle POR\rvert$	(angles in any quadrilateral add up to 360°)
But $\lvert\angle PBR\rvert = 180° - (\lvert\angle CAB\rvert + \lvert\angle CBA\rvert)$	(angles in a triangle)
So $\lvert\angle POR\rvert = \lvert\angle CAB\rvert + \lvert\angle CBA\rvert$	
But $\lvert\angle PQR\rvert = \frac{1}{2}\lvert\angle POR\rvert$	
So $\lvert\angle PQR\rvert = \dfrac{1}{2}(\lvert\angle CAB\rvert + \lvert\angle CBA\rvert)$	

OR

Let $OA \cap PQ = \{D\}$	
$\lvert OP\rvert = \lvert OQ\rvert \Rightarrow \lvert AP\rvert = \lvert AQ\rvert$	(Pythagoras)
$\lvert\angle PAD\rvert = \lvert\angle QAD\rvert$	(bisector)
$\therefore \Delta PDA \equiv \Delta QDA$	(S.A.S.)
$\therefore \lvert\angle PDA\rvert = \lvert\angle QDA\rvert = 90°$	
$\lvert\angle DAQ\rvert = 90° - \lvert\angle DQA\rvert$	
$= \lvert\angle OQD\rvert$	
$\therefore \lvert\angle PAQ\rvert = 2\lvert\angle OQD\rvert$	
Similarly, $\lvert\angle RBQ\rvert = 2\lvert OQR\rvert$	
Adding these two gives the required result.	

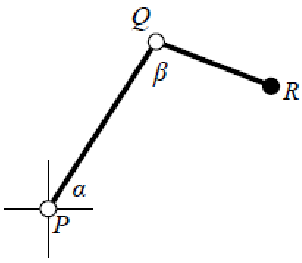
OR

Let $OA \cap PQ = \{D\}$	
$\lvert OP\rvert = \lvert OQ\rvert \Rightarrow \lvert AP\rvert = \lvert AQ\rvert$	(Pythagoras)
$\lvert\angle APQ\rvert = \lvert\angle AQP\rvert$	(isosceles triangle theorem)
Similarly, $\lvert\angle RQB\rvert = \lvert\angle RBQ\rvert$	
$\lvert\angle AQP\rvert + \lvert\angle PQR\rvert + \lvert\angle RQB\rvert = 180^0$	
$\lvert\angle PQR\rvert = 180^0 - \lvert\angle AQP\rvert - \lvert\angle RQB\rvert$	
$\lvert\angle CAB\rvert = 180^0 - 2\lvert\angle AQP\rvert$	
$\lvert\angle CBA\rvert = 180^0 - 2\lvert\angle RQB\rvert$	
$\Rightarrow \lvert\angle CAB\rvert + \lvert\angle CBA\rvert = 360^0 - 2[\lvert\angle AQP\rvert + \lvert\angle RQB\rvert]$	
$\Rightarrow \dfrac{1}{2}[\lvert\angle CAB\rvert + \lvert\angle CBA\rvert] = 180^0 - \lvert\angle AQP\rvert - \lvert\angle RQB\rvert = \lvert\angle PQR\rvert$	

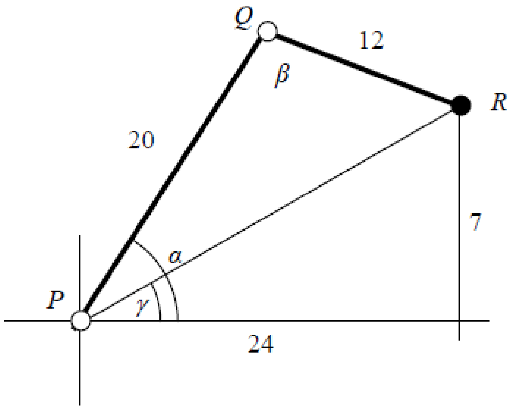
Q8

Question 8 **(75 marks)**

The diagram is a representation of a robotic arm that can move in a vertical plane. The point P is fixed, and so are the lengths of the two segments of the arm. The controller can vary the angles α and β from 0° to 180°.



(a) Given that $|PQ| = 20$ cm and $|QR| = 12$ cm, determine the values of the angles α and β so as to locate R, the tip of the arm, at a point that is 24 cm to the right of P, and 7 cm higher than P. Give your answers correct to the nearest degree.



$$|PR|^2 = 7^2 + 24^2$$
$$|PR| = 25$$

$$25^2 = 20^2 + 12^2 - 2(20)(12)\cos\beta$$
$$\cos\beta = -0{\cdot}16875$$
$$\beta \approx 100°$$

$$12^2 = 25^2 + 20^2 - 2(25)(20)\cos(\alpha - \gamma)$$
$$\cos(\alpha - \gamma) = 0{\cdot}881$$
$$\alpha - \gamma \approx 28{\cdot}237°$$

$$\tan\gamma = \frac{7}{24}$$
$$\gamma \approx 16{\cdot}260°$$

$$\therefore \alpha \approx 44°$$

(b) In setting the arm to the position described in part **(a)**, which will cause the greater error in the location of R: an error of $1°$ in the value of α or an error of $1°$ in the value of β?

Justify your answer. You may assume that if a point moves along a circle through a small angle, then its distance from its starting point is equal to the length of the arc travelled.

Ans: α

Reason: $1°$ error in α causes R to move along an arc of radius 25.
$1°$ error in β causes R to move along an arc of radius 12.

So, since $l = r\theta$, and θ is the same in each case, the point moves further in the first case.

(c) The answer to part **(b)** above depends on the particular position of the arm. That is, in certain positions, the location of R is more sensitive to small errors in α than to small errors in β, while in other positions, the reverse is true. Describe, with justification, the conditions under which each of these two situations arises.

More sensitive to errors in α when $|PR| > 12$

More sensitive to errors in β when $|PR| < 12$

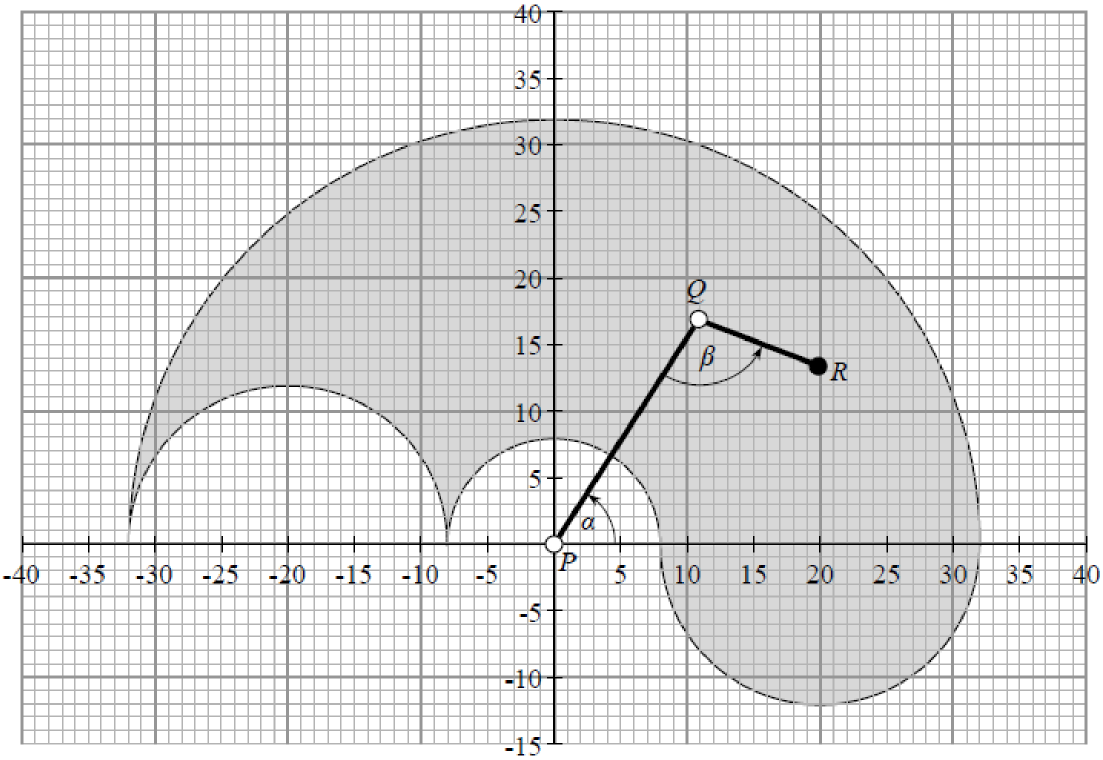
The condition $|PR| > 12$

is true whenever

$$\beta > \cos^{-1}\left(\frac{5}{6}\right) \approx 33{\cdot}6°$$

(Borderline case is when ΔPQR is isosceles with $|QR| = |RP|$.)

(d) Illustrate the set of all possible locations of the point R on the coordinate diagram below. Take P as the origin and take each unit in the diagram to represent a centimetre in reality. Note that α and β can vary only from 0° to 180°.

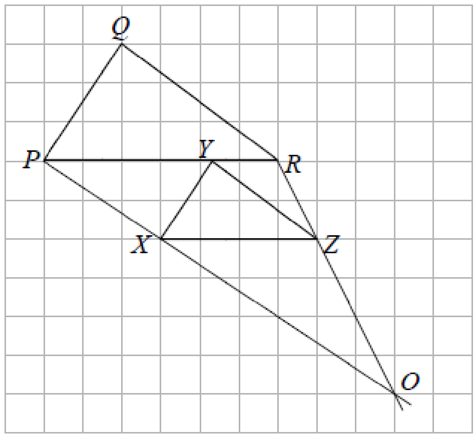


2011

Paper 2 – Project Maths – Section A – Q4

Question 4 **(25 marks)**

Two triangles are drawn on a square grid as shown. The points P, Q, R, X, and Z are on vertices of the grid, and the point Y lies on $[PR]$. The triangle PQR is an enlargement of the triangle XYZ.



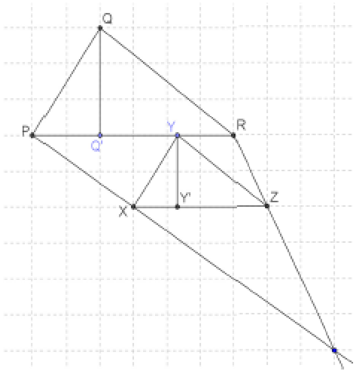
(a) Calculate the scale factor of the enlargement, showing your work.

$$\frac{|PR|}{|XZ|}=\frac{6}{4}=\frac{3}{2}$$

(b) By construction or otherwise, locate the centre of enlargement on the diagram above.

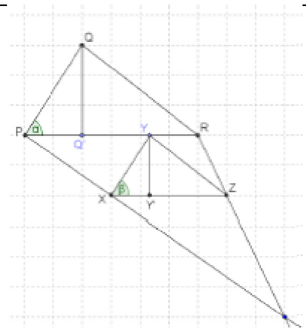
Shown as O above.

(c) Calculate $|YR|$ in grid units.



$$|Y'Z|=\frac{2}{3}|Q'R|=\frac{2}{3}(4)=\frac{8}{3}$$

$$|YR|=\frac{8}{3}-1=\frac{5}{3}$$



$$\tan\alpha=\frac{3}{2}$$

$$\alpha=\beta$$

$$\frac{2}{|XY'|}=\frac{3}{2}$$

$$|XY'|=\frac{4}{3}$$

$$|YR|=3-\frac{4}{3}=\frac{5}{3}$$