Induction

Maths Past Exam Questions

Marking Schemes

Higher Level

2012

Paper 1 – Project Maths – Section Q4

Question 4 **(25 marks)**

(a) Prove, by induction, the formula for the sum of the first n terms of a geometric series. That is, prove that, for $r \neq 1$:

$$a + ar + ar^2 + \cdots + ar^{n-1} = \frac{a(1-r^n)}{1-r}.$$

$P(n)$: $\quad a + ar + ar^2 + \cdots + ar^{n-1} = \dfrac{a(1-r^n)}{1-r}$

Check $P(1)$: $\quad a = \dfrac{a(1-r)}{1-r}$, which is true.

Assume $P(k)$: $\quad a + ar + ar^2 + \cdots + ar^{k-1} = \dfrac{a(1-r^k)}{1-r}$

Then: $\quad \underbrace{a + ar + ar^2 + \cdots + ar^{k-1}} + ar^k$

$$= \frac{a(1-r^k)}{1-r} + ar^k$$

$$= \frac{a(1-r^k) + ar^k(1-r)}{1-r}$$

$$= \frac{a(1-\cancel{r^k}+\cancel{r^k}-r^{k+1})}{1-r}$$

$$= \frac{a(1-r^{k+1})}{1-r}$$

which establishes $P(k+1)$.

Since we have $P(1) \wedge \{\forall k \in \mathbb{N}, (P(k) \Rightarrow P(k+1)\}$, it follows that $P(n)$ holds $\forall n \in \mathbb{N}$.

(b) By writing the recurring part as an infinite geometric series, express the following number as a fraction of integers:

$$5{\cdot}\dot{2}\dot{1} \quad = \quad 5{\cdot}2121212121\ldots$$

$$5{\cdot}\dot{2}\dot{1} = 5 + \frac{21}{100} + \frac{21}{10000} + \frac{21}{1000000} + \cdots$$

$= 5 +$ [geometric series with $a = \frac{21}{100}$, $\quad r = \frac{1}{100}$].

$$= 5 + \frac{\frac{21}{100}}{1-\frac{1}{100}} = 5 + \frac{21}{100-1} = 5\tfrac{21}{99} = 5\tfrac{7}{33}.$$