Sequences and Series

Maths Past Exam Questions

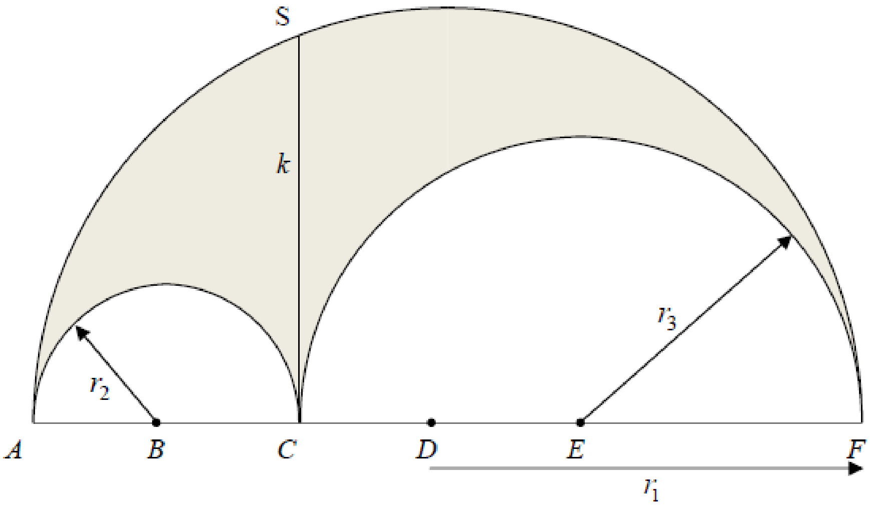
Marking Schemes

Higher Level

2013

Paper 1 – Project Maths – Section B Q9

(b) The shaded region in the diagram below is called an **arbelos.** It is a plane semicircular region of radius r_1 from which semicircles of radius r_2 and r_3 are removed, as shown. In the diagram $SC \perp AF$ and $|SC| = k$.



(i) Show that, for fixed r_1, the perimeter of the arbelos is independent of the values of r_2 and r_3.

$$\text{Perimeter} = \pi r_1 + (\pi r_2 + \pi r_3) = \pi(r_1 + (r_2 + r_3)) = \pi(r_1 + r_1) = 2\pi r_1$$

which is independent of r_2 and r_3

(ii) If $r_2 = 2$ and $r_3 = 4$, show that the area of the arbelos is the same as the area of the circle of diameter k.

$$\begin{aligned}\text{Area of arbelos} &= \tfrac{1}{2}\pi r_1^2 - \tfrac{1}{2}\pi(r_2^2 + r_3^2) \\ &= \tfrac{1}{2}\pi(6^2) - \tfrac{1}{2}\pi(2^2 + 4^2) \\ &= \tfrac{1}{2}\pi(36 - 20) \\ &= 8\pi\end{aligned}$$

$k^2 + 4 = 36$

$k = \sqrt{32}$

$$\text{Area of circle} = \pi\left(\frac{k}{2}\right)^2 = \pi\left(\frac{\sqrt{32}}{2}\right)^2 = \frac{\pi\left(\sqrt{32}\right)^2}{4} = 8\pi$$

(c) To investigate the area of an arbelos, a student fixed the value of r_1 at 6 cm and completed the following table for different values of r_2 and r_3.

(i) Complete the table.

r_1	r_2	r_3	Area of arbelos
6	1	5	$\frac{1}{2}\pi\left(6^2-(1^2+5^2)\right)=5\pi\text{ cm}^2$
6	2	4	$\frac{1}{2}\pi\left(6^2-(2^2+4^2)\right)=8\pi\text{ cm}^2$
6	3	3	$\frac{1}{2}\pi\left(6^2-(3^2+3^2)\right)=9\pi\text{ cm}^2$
6	4	2	$\frac{1}{2}\pi\left(6^2-(4^2+1^2)\right)=8\pi\text{ cm}^2$
6	5	1	$\frac{1}{2}\pi\left(6^2-(5^2+1^2)\right)=5\pi\text{ cm}^2$

(ii) In general for $r_1 = 6$ cm and $r_2 = x$, $0 < x < 6, x \in \mathbb{R}$, find an expression in x for the area of the arbelos.

$$\begin{aligned}\text{Area of arbelos } &= \tfrac{1}{2}\pi r_1^2 - \tfrac{1}{2}\pi\left(r_2^2 + r_3^2\right)\\ &= \tfrac{1}{2}\pi\left(r_1^2 - (r_2^2 + r_3^2\right)\\ &= \tfrac{1}{2}\pi\left(36 - \left(x^2 + (6-x)^2\right)\right)\\ &= \pi\left(6x - x^2\right)cm^2\end{aligned}$$

(iii) Hence, or otherwise, find the maximum area of an arbelos that can be formed in a semi circle of radius 6 cm.

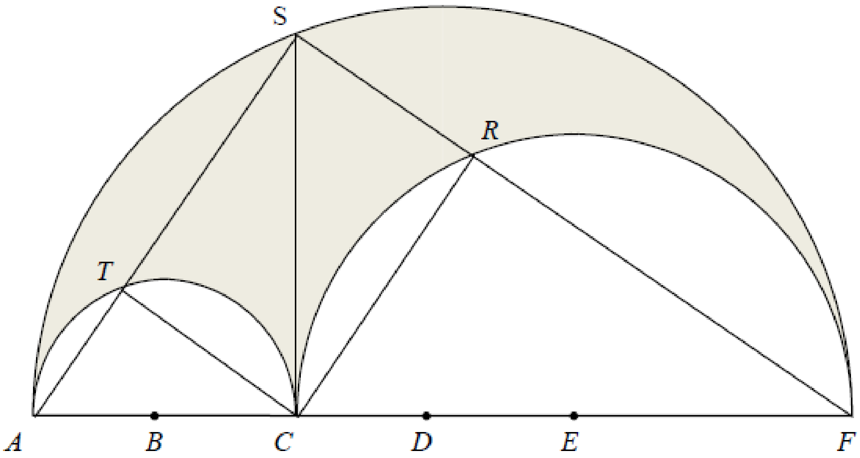
$$A = \pi\left(6x - x^2\right) \;\Rightarrow\; \frac{dA}{dx} = \pi(6-2x)$$

$$\pi(6-2x) = 0 \;\Rightarrow\; x = 3$$

$$\frac{dA}{dx} = \pi(6-2x) \;\Rightarrow\; \frac{d^2A}{dx^2} = -2\pi < 0 \;\Rightarrow\; \text{maximum}$$

Maximum area when $x = 3$, giving area = 9π cm^2

(d) AS and FS cut the two smaller semicircles at T and R respectively.
Prove that $RSTC$ is a rectangle.



$|\angle TSR| = 90°$ ..... Angle in a semicircle
$|\angle CTA| = 90°$ ..... Angle in a semicircle
Hence, $|\angle STC| = 90°$
$|\angle FRC| = 90°$ ..... Angle in a semicircle
Hence, $|\angle CRS| = 90°$

Hence, the angles in $RSTC$ are right angles and so $RSTC$ is a rectangle.