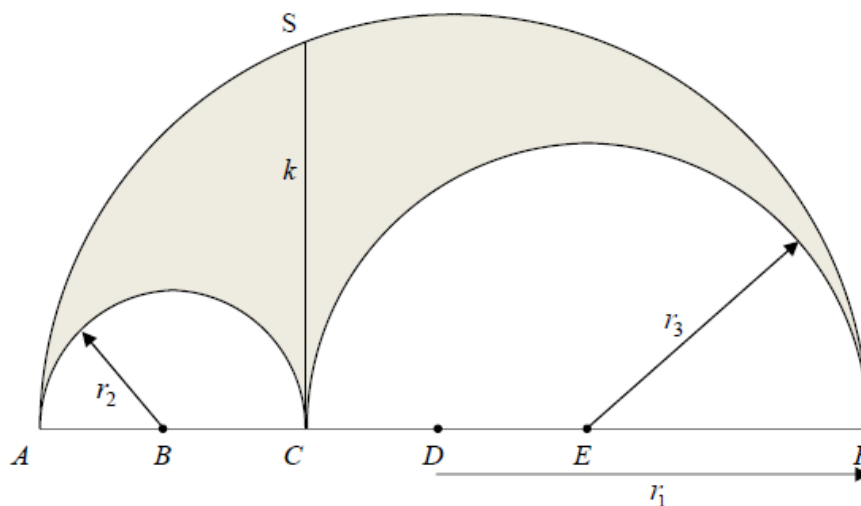




Sequences and Series
Maths Past Exam Questions
Marking Schemes
Higher Level

Paper 1 – Project Maths – Section B Q9

- (b) The shaded region in the diagram below is called an **arbelos**. It is a plane semicircular region of radius r_1 from which semicircles of radius r_2 and r_3 are removed, as shown. In the diagram $SC \perp AF$ and $|SC|=k$.



- (i) Show that, for fixed r_1 , the perimeter of the arbelos is independent of the values of r_2 and r_3 .

$$\text{Perimeter} = \pi r_1 + (\pi r_2 + \pi r_3) = \pi(r_1 + r_2 + r_3) = \pi(r_1 + r_1) = 2\pi r_1$$

which is independent of r_2 and r_3

- (ii) If $r_2 = 2$ and $r_3 = 4$, show that the area of the arbelos is the same as the area of the circle of diameter k .

$$\begin{aligned} \text{Area of arbelos} &= \frac{1}{2}\pi r_1^2 - \frac{1}{2}\pi(r_2^2 + r_3^2) \\ &= \frac{1}{2}\pi(6^2) - \frac{1}{2}\pi(2^2 + 4^2) \\ &= \frac{1}{2}\pi(36 - 20) \\ &= 8\pi \end{aligned}$$

$$k^2 + 4 = 36$$

$$k = \sqrt{32}$$

$$\text{Area of circle} = \pi\left(\frac{k}{2}\right)^2 = \pi\left(\frac{\sqrt{32}}{2}\right)^2 = \frac{\pi(\sqrt{32})^2}{4} = 8\pi$$

- (c) To investigate the area of an arbelos, a student fixed the value of r_1 at 6 cm and completed the following table for different values of r_2 and r_3 .

(i) Complete the table.

r_1	r_2	r_3	Area of arbelos
6	1	5	$\frac{1}{2}\pi(6^2 - (1^2 + 5^2)) = 5\pi \text{ cm}^2$
6	2	4	$\frac{1}{2}\pi(6^2 - (2^2 + 4^2)) = 8\pi \text{ cm}^2$
6	3	3	$\frac{1}{2}\pi(6^2 - (3^2 + 3^2)) = 9\pi \text{ cm}^2$
6	4	2	$\frac{1}{2}\pi(6^2 - (4^2 + 1^2)) = 8\pi \text{ cm}^2$
6	5	1	$\frac{1}{2}\pi(6^2 - (5^2 + 1^2)) = 5\pi \text{ cm}^2$

- (ii) In general for $r_1 = 6 \text{ cm}$ and $r_2 = x$, $0 < x < 6$, $x \in \mathbb{R}$, find an expression in x for the area of the arbelos.

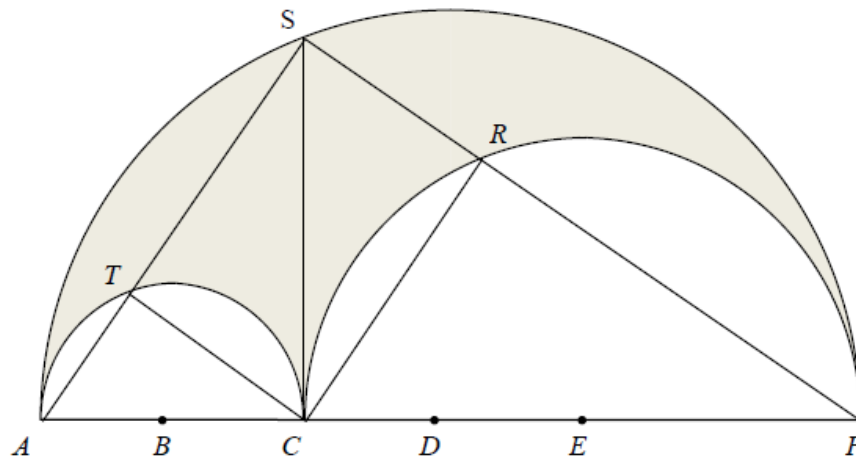
$$\begin{aligned}
 \text{Area of arbelos} &= \frac{1}{2}\pi r_1^2 - \frac{1}{2}\pi(r_2^2 + r_3^2) \\
 &= \frac{1}{2}\pi(r_1^2 - (r_2^2 + r_3^2)) \\
 &= \frac{1}{2}\pi(36 - (x^2 + (6-x)^2)) \\
 &= \pi(6x - x^2) \text{ cm}^2
 \end{aligned}$$

- (iii) Hence, or otherwise, find the maximum area of an arbelos that can be formed in a semi circle of radius 6 cm.

$$\begin{aligned}
 A = \pi(6x - x^2) &\Rightarrow \frac{dA}{dx} = \pi(6 - 2x) \\
 \pi(6 - 2x) = 0 &\Rightarrow x = 3 \\
 \frac{dA}{dx} = \pi(6 - 2x) &\Rightarrow \frac{d^2A}{dx^2} = -2\pi < 0 \Rightarrow \text{maximum}
 \end{aligned}$$

Maximum area when $x = 3$, giving area = $9\pi \text{ cm}^2$

- (d) AS and FS cut the two smaller semicircles at T and R respectively.
Prove that $RSTC$ is a rectangle.



$$|\angle TSR| = 90^\circ \quad \dots \text{Angle in a semicircle}$$

$$|\angle CTA| = 90^\circ \quad \dots \text{Angle in a semicircle}$$

$$\text{Hence, } |\angle STC| = 90^\circ$$

$$|\angle FRC| = 90^\circ \quad \dots \text{Angle in a semicircle}$$

$$\text{Hence, } |\angle CRS| = 90^\circ$$

Hence, the angles in $RSTC$ are right angles and so $RSTC$ is a rectangle.