The Circle

Maths Past Exam Questions

Marking Schemes

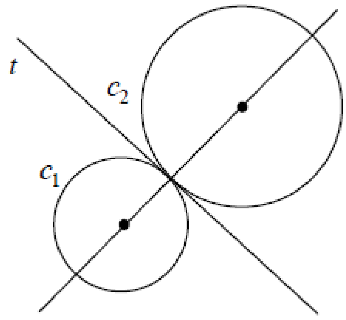
Higher Level

2013

Paper 2 – Project Maths – Section A Q4

Question 4 **(25 marks)**

The circles c_1 and c_2 touch externally as shown.



$$\sqrt{g^2 + f^2 - c} = \sqrt{1+1+7} = 3$$

(a) Complete the following table:

Circle	Centre	Radius	Equation
c_1	$(-3, -2)$	2	$(x+3)^2 + (y+2)^2 = 4$ **OR** $x^2 + y^2 + 6x + 4y + 9 = 0$
c_2	$(1, 1)$	3	$x^2 + y^2 - 2x - 2y - 7 = 0$

(b) **(i)** Find the co-ordinates of the point of contact of c_1 and c_2.

Divide line segment joining $(-3, -2)$ and $(1, 1)$ in ratio 2 : 3

$$\left(\frac{2(1)+3(-3)}{2+3}, \frac{2(1)+3(-2)}{2+3}\right) = \left(-\frac{7}{5}, -\frac{4}{5}\right)$$

OR

Slope line of centres $= \frac{3}{4}$.

Equation line of centres: $y - 1 = \frac{3}{4}(x-1) \Rightarrow 3x - 4y + 1 = 0$

$c_1 - c_2 = 4x + 3y + 8 = 0$

$4x + 3y + 8 = 0 \cap 3x - 4y + 1 = 0 \Rightarrow x = -\frac{7}{5}, y = -\frac{4}{5}$

(ii) Hence, or otherwise, find the equation of the tangent, t, common to c_1 and c_2.

Slope of line of centres: $\frac{1+2}{1+3}=\frac{3}{4}$

Slope of tangent: $m=-\frac{4}{3}$

Equation of tangent: $y+\frac{4}{5}=-\frac{4}{3}\left(x+\frac{7}{5}\right)$

$\Rightarrow\ 3y+\frac{12}{5}=-4x-\frac{28}{5}$

$\Rightarrow\ 4x+3y+8=0$

OR

$c_1-c_2=x^2+y^2+6x+4y+9-(x^2+y^2-2x-2y-7)=0$

$\Rightarrow 6x+4y+9-(-2x-2y-7)=0$

$\Rightarrow 8x+6y+16=0 \Rightarrow 4x+3y+8=0$

OR

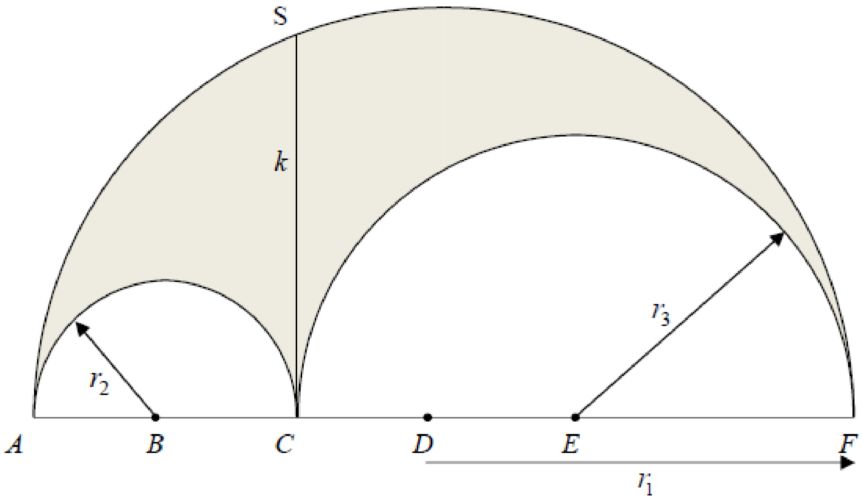
$xx_1+yy_1+g(x+x_1)+f(y+y_1)+c=0$

$x(-\frac{7}{5})+y(-\frac{4}{5})+3(x+(-\frac{7}{5}))+2(y+(-\frac{4}{5})+9=0$

$\Rightarrow 4x+3y+8=0$

Paper 2 – Project Maths – Section B Q9 B-D

(b) The shaded region in the diagram below is called an **arbelos.** It is a plane semicircular region of radius r_1 from which semicircles of radius r_2 and r_3 are removed, as shown. In the diagram $SC \perp AF$ and $|SC| = k$.



(i) Show that, for fixed r_1, the perimeter of the arbelos is independent of the values of r_2 and r_3.

Perimeter $= \pi r_1 + (\pi r_2 + \pi r_3) = \pi(r_1 + (r_2 + r_3)) = \pi(r_1 + r_1) = 2\pi r_1$

which is independent of r_2 and r_3

(ii) If $r_2 = 2$ and $r_3 = 4$, show that the area of the arbelos is the same as the area of the circle of diameter k.

$$\begin{aligned} \text{Area of arbelos } &= \tfrac{1}{2}\pi r_1^2 - \tfrac{1}{2}\pi(r_2^2 + r_3^2) \\ &= \tfrac{1}{2}\pi(6^2) - \tfrac{1}{2}\pi(2^2 + 4^2) \\ &= \tfrac{1}{2}\pi(36 - 20) \\ &= 8\pi \end{aligned}$$

$k^2 + 4 = 36$

$k = \sqrt{32}$

Area of circle $= \pi\left(\frac{k}{2}\right)^2 = \pi\left(\frac{\sqrt{32}}{2}\right)^2 = \frac{\pi(\sqrt{32})^2}{4} = 8\pi$

(c) To investigate the area of an arbelos, a student fixed the value of r_1 at 6 cm and completed the following table for different values of r_2 and r_3.

(i) Complete the table.

r_1	r_2	r_3	Area of arbelos
6	1	5	$\frac{1}{2}\pi\left(6^2-(1^2+5^2)\right)=5\pi\text{ cm}^2$
6	2	4	$\frac{1}{2}\pi\left(6^2-(2^2+4^2)\right)=8\pi\text{ cm}^2$
6	3	3	$\frac{1}{2}\pi\left(6^2-(3^2+3^2)\right)=9\pi\text{ cm}^2$
6	4	2	$\frac{1}{2}\pi\left(6^2-(4^2+1^2)\right)=8\pi\text{ cm}^2$
6	5	1	$\frac{1}{2}\pi\left(6^2-(5^2+1^2)\right)=5\pi\text{ cm}^2$

(ii) In general for $r_1 = 6$ cm and $r_2 = x$, $0 < x < 6, x \in \mathbb{R}$, find an expression in x for the area of the arbelos.

$$\begin{aligned}\text{Area of arbelos } &= \tfrac{1}{2}\pi r_1^2 - \tfrac{1}{2}\pi\left(r_2^2+r_3^2\right)\\ &= \tfrac{1}{2}\pi\left(r_1^2-(r_2^2+r_3^2)\right)\\ &= \tfrac{1}{2}\pi\left(36-\left(x^2+(6-x)^2\right)\right)\\ &= \pi\left(6x-x^2\right)cm^2\end{aligned}$$

(iii) Hence, or otherwise, find the maximum area of an arbelos that can be formed in a semi circle of radius 6 cm.

$$A=\pi\left(6x-x^2\right) \Rightarrow \frac{dA}{dx}=\pi(6-2x)$$

$$\pi(6-2x)=0 \Rightarrow x=3$$

$$\frac{dA}{dx}=\pi(6-2x) \Rightarrow \frac{d^2A}{dx^2}=-2\pi<0 \Rightarrow \text{maximum}$$

Maximum area when $x = 3$, giving area = $9\pi\text{ cm}^2$

(d) *AS* and *FS* cut the two smaller semicircles at *T* and *R* respectively.
Prove that *RSTC* is a rectangle.

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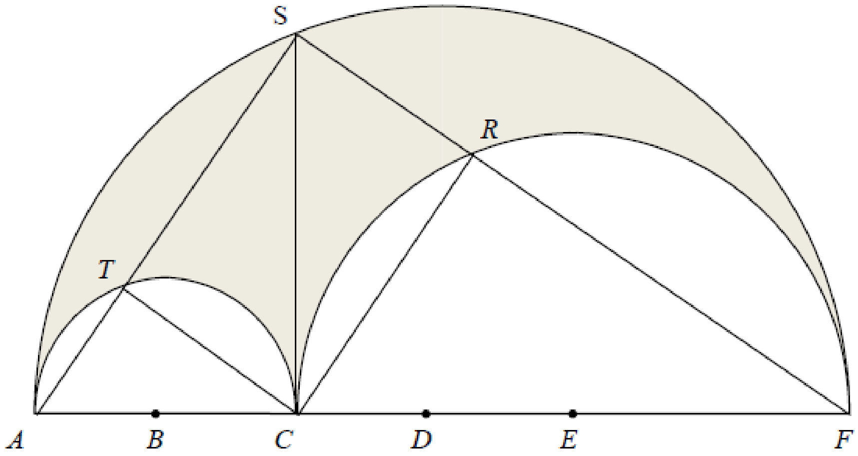


$|\angle TSR| = 90°$ ..... Angle in a semicircle
$|\angle CTA| = 90°$ ..... Angle in a semicircle
Hence, $|\angle STC| = 90°$
$|\angle FRC| = 90°$ ..... Angle in a semicircle
Hence, $|\angle CRS| = 90°$

Hence, the angles in *RSTC* are right angles and so *RSTC* is a rectangle.

2012

Paper 2 – Project Maths – Section A Q2

Question 2 **(25 marks)**

The equations of two circles are:

$$c_1 : x^2 + y^2 - 6x - 10y + 29 = 0$$
$$c_2 : x^2 + y^2 - 2x - 2y - 43 = 0$$

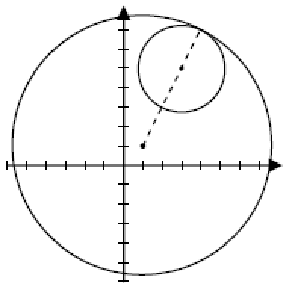
(a) Write down the centre and radius-length of each circle.

$c_1 : (x-3)^2 + (y-5)^2 = 5$ $\therefore$ centre (3, 5); radius $\sqrt{5}$.	$c_2 : (x-1)^2 + (y-1)^2 = 45$ $\therefore$ centre (1, 1); radius $\sqrt{45} = 3\sqrt{5}$.

(b) Prove that the circles are touching.

Distance between centres: $\sqrt{(3-1)^2 + (5-1)^2} = \sqrt{2^2 + 4^2} = \sqrt{20} = 2\sqrt{5}$

The distance between the centres is the difference of the radii $\Rightarrow$ circles touch (internally).



(c) Verify that (4, 7) is the point that they have in common.

$4^2+7^2-6(4)-10(7)+29=0 \quad \Rightarrow \quad (4,7)\in c_1$

$4^2+7^2-2(4)-2(7)-43=0 \quad \Rightarrow \quad (4,7)\in c_2$

OR

$c_1-c_2: \quad x+2y-18=0 \Rightarrow x=-2y+18$

$(-2y+18)^2+y^2-6(-2y+18))-10y+29=0$

$(y-7)^2=0$

$y=7$

$x=4$

$\therefore$ (4, 7) common

(d) Find the equation of the common tangent.

Slope from (3, 5) to (4, 7) is: $\dfrac{7-5}{4-3}=2$

$\therefore$ slope of tangent $=-\frac{1}{2}$.

Equation of tangent:

$$y-7=-\tfrac{1}{2}(x-4)$$
$$2y-14=-x+4$$
$$x+2y-18=0$$

OR

Equation of Tangent: $c_1-c_2: x+2y-18=0$

OR

$$(x-h)(x_1-h)+(y-k)(y_1-k)=r^2$$
$$(x-3)(4-3)+(y-5)(7-5)=\left(\sqrt{5}\right)^2$$
$$(x-3)+(y-5)(2)=5$$
$$x+2y-18=0$$

OR

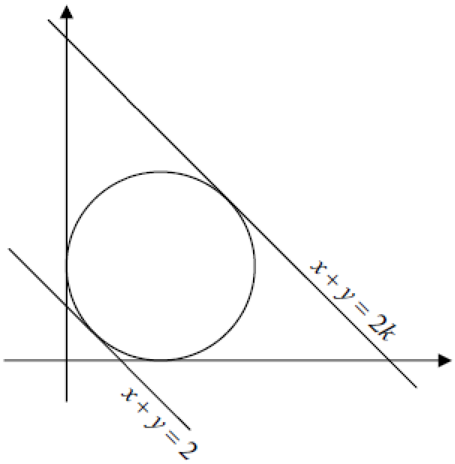
$$xx_1+yy_1+g(x+x_1)+f(y+y_1)+c=0$$
$$4x+7y-3(x+4)-5(y+7)+29=0$$
$$x+2y-18=0$$

Paper 2 – Project Maths – Section A Q3

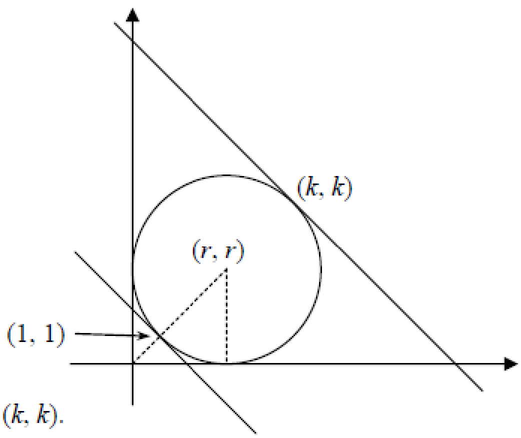
Question 3 **(25 marks)**

The circle shown in the diagram has, as tangents, the x-axis, the y-axis, the line $x + y = 2$ and the line $x + y = 2k$, where $k > 1$.

Find the value of k.



$$r^2 + r^2 = \left(r + \sqrt{2}\right)^2$$
$$2r^2 = r^2 + 2\sqrt{2}r + 2$$
$$r^2 - 2\sqrt{2}r - 2 = 0$$
$$\left(r - \sqrt{2}\right)^2 = 4$$
$$r = \sqrt{2} + 2, \quad (r > 0)$$



(r, r) is midpoint of segment from $(1, 1)$ to (k, k).

$$\frac{k+1}{2} = r$$
$$k = 2r - 1$$
$$k = 3 + 2\sqrt{2}$$

OR

Equation of circle: $(x-r)^2+(y-r)^2=r^2$

The line $x+y=2$ intersects the circle at one point only.

$$y=2-x \Rightarrow (x-r)^2+((2-x)-r)^2=r^2$$

$$\Rightarrow x^2+(2-x)^2+r^2-4r=0$$

$$\Rightarrow 2x^2-4x+(r^2-4r+4)=0$$

One real root $\Rightarrow b^2-4ac=0$

$$\Rightarrow 16-4(2)(r^2-4r+4)=0$$

$$\Rightarrow r=\frac{4\pm\sqrt{16-8}}{2}=2\pm\sqrt{2}$$

But $2-\sqrt{2}$ is too small, so $r=2+\sqrt{2}$

$(1, 1) \rightarrow (2+\sqrt{2},\ 2+\sqrt{2}) \rightarrow (3+2\sqrt{2},\ 3+2\sqrt{2}) = (k, k)$

OR

Centre (r, r)

Perpendicular distance to $x+y-2=0$ equals radius, r.

$$\left|\frac{r+r-2}{\sqrt{2}}\right|=r$$

$$\Rightarrow 2r-2=\pm r\sqrt{2}$$

$$r=\frac{2}{2\pm\sqrt{2}}=2\mp\sqrt{2}$$

But $2-\sqrt{2}$ is too small, so $r=2+\sqrt{2}$

$(1, 1) \rightarrow (2+\sqrt{2},\ 2+\sqrt{2}) \rightarrow (3+2\sqrt{2},\ 3+2\sqrt{2}) = (k, k)$

OR

Having found r as above, find k by setting perpendicular distance from centre (r, r) to $x+y-2k=0$ equal to r:

$$\left|\frac{r+r-2k}{\sqrt{2}}\right|=r$$

$$\Rightarrow 2r-2k=\pm\sqrt{2}r$$

$$\Rightarrow 2\left(2+\sqrt{2}\right)-2k=\pm\sqrt{2}\left(2+\sqrt{2}\right)$$

$$\Rightarrow 4+2\sqrt{2}-2k=\pm\left(2\sqrt{2}+2\right)$$

$$\Rightarrow 2k=4+2\sqrt{2}\pm\left(2\sqrt{2}+2\right)$$

$$\Rightarrow k=2+\sqrt{2}\pm\left(\sqrt{2}+1\right)$$

$$\Rightarrow k=3+2\sqrt{2} \text{ or } 1.$$

$k=1$ corresponds to the lower line, so the answer is $k=3+2\sqrt{2}$

2011

Paper 2 – Project maths – Q5

Question 5 **(25 marks)**

The line $x+3y=20$ intersects the circle $x^2+y^2-6x-8y=0$ at the points P and Q.

Find the equation of the circle that has $[PQ]$ as diameter.

Line $\Rightarrow$ $x=20-3y$

$$\therefore (20-3y)^2+y^2-6(20-3y)-8y=0$$

$$9y^2-120y+400+y^2-120+18y-8y=0$$
$$10y^2-110y+280=0$$
$$y^2-11y+28=0$$
$$(y-7)(y-4)=0$$
$$y=7 \quad \text{or} \quad y=4$$
$$x=-1 \quad \text{or} \quad x=8$$

$P(-1, 7)$ and $Q(8, 4)$

Centre is midpoint of $[PQ]$: $C\left(\frac{7}{2}, \frac{11}{2}\right)$

$$r=\sqrt{\left(\frac{7}{2}+1\right)^2+\left(\frac{11}{2}-7\right)^2}$$

$$=\sqrt{20{\cdot}25+2{\cdot}25}$$

$$=\sqrt{22{\cdot}5} \quad \text{or} \quad \sqrt{\frac{45}{2}}$$

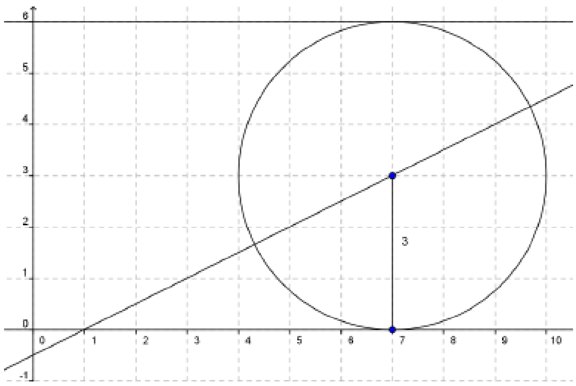
Equation: $\left(x-\frac{7}{2}\right)^2+\left(y-\frac{11}{2}\right)^2=\frac{45}{2}$

2010

Paper 2 – Project Maths – Q4

Question 4 **(25 marks)**

(a) The centre of a circle lies on the line $x - 2y - 1 = 0$. The x-axis and the line $y = 6$ are tangents to the circle. Find the equation of this circle.



$$r = 3$$
$$\text{centre: } (h, 3)$$
$$h - 2(3) - 1 = 0$$
$$h = 7$$

Equation of circle:

$$(x-7)^2 + (y-3)^2 = 3^2$$
$$(x-7)^2 + (y-3)^2 = 9$$

or

$$x^2 + y^2 - 14x - 6y + 49 = 0$$

(b) A different circle has equation $x^2 + y^2 - 6x - 12y + 41 = 0$. Show that this circle and the circle in part **(a)** touch externally.

$x^2 + y^2 - 6x - 12y + 41 = 0$.

centre: (3, 6); radius= $\sqrt{9 + 36 - 41} = \sqrt{4} = 2$.

Distance between centres: $\sqrt{(7-3)^2 + (3-6)^2} = \sqrt{25} = 5$

Sum of radii: 3 + 2 = 5 = distance between centres.

$\therefore$ circles touch externally.